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On the Forms of Sextic Scrolls Having no Rectilinear Directrix.

BY VIRGIL SNYDER.

In my papers on sextic scrolls which appeared in Volume XXV of the Journal, two families of scrolls were omitted, namely, those whose generators are bisecants of a twisted cubic or quartic curve and those having a tacnodal directrix curve, analogous to those having a tacnodal directrix line, types 58-66, in my paper in the Journal, Volume XXVII, p. 77.

The present paper will discuss these two types and supply various other minor omissions of the previous papers. The list of types here found, together with those published in the preceding papers, is believed to be complete. The method will be the same as that employed before, namely, the establishment of a multiple correspondence between the points of two curves and drawing a generator between them.

§1.— S_6 Having Two Distinct Double Conics.

1. In type XIX (p. 90, Vol. XXV), the cubic is a plane nodal cubic, and all three conics pass through two common points. If in the correspondence there defined, $\mu = 0$ corresponds to 0 and ∞ , and if $\lambda = 0$ corresponds to 0 and ∞ , the line of intersection of the planes of the two conics becomes a double generator. The residual nodal curve is now a twisted quartic. The symbol is $2c_2^2 + c_4^2 + g^2$ (1). If the (2,2) correspondence has a double element, the surface becomes unicursal and has two double generators: $2c_2^2 + c_4^2 + 2g^2$ (2). These two forms are self-dual. The correspondence given in the Bulletin of the American Math. Society, Volume IX, p. 240, defines the S_6 of genus 1, having the symbol $c_{3,2}^2 + 3c_2^2$. The

three conics have two points in common. If, in this correspondence, a double element be introduced, a double generator appears; the symbol becomes $3c_2^2 + c_{3,2}^2 + g^2$. (Type LVIII, Vol. XXV, p. 81). If the plane determined by g^2 and the tangent to c_1 at their point of intersection also contains the tangent to c_2 at its point of intersection with g^2 , then the latter counts for two double generators and the residual nodal curve is a fourth double conic, cutting $\overline{2g^2}$ twice and each of the preceding conics once (3). $4c_2^2 + \overline{2g^2}$.

2. Any two conics which touch each other but lie in different planes may be written in the form:

$$\begin{aligned} c_1: \quad x &= 1, \quad y = \mu, \quad z = 0, \quad w = \mu^2, \\ c_2: \quad x &= 0, \quad y = \lambda, \quad z = 1, \quad w = \lambda^2. \end{aligned}$$

The equations of a line joining λ to μ are

$$\mu x + \lambda z - y = 0, \quad \mu^2 x + \lambda^2 z - w = 0.$$

When the form of the correspondence between λ, μ is

$$c\lambda^2 + d\mu^2 + c\lambda\mu + f\lambda + g\mu + h = 0$$

the surface is an S_6 having the common tangent $x = 0, g = 0$ for tacnodal generator. The equation is

$$\begin{vmatrix} cx - dz - ez & ey + fx - gz & dw + gy + hx & 0 \\ 0 & cx - dz - cz & ey + fx - gz & dw + gy + hx \\ z^2 + zx & -2yz & y^2 - xw & 0 \\ 0 & z^2 + zx & -2yz & y^2 - xw \end{vmatrix} = 0.$$

The plane $cx - dz = 0$ contains x^4 and a simple conic. The residual nodal curve is a nodal quartic, the node being at the point of tangency of the double conics. $2c_2^2 + c_{4,2}^2 + \overline{2g^2}$ (4). This surface belongs to a linear complex when $e = 0$.

3. Given two conics $c_1(\lambda)$ and $c_2(\mu)$ which intersect in one point, $(0, 0)$. Let λ, μ be in $(2, 2)$ correspondence with $(0, 0)$ as self-corresponding double point. Let

$$\begin{aligned} c_1: \quad x &= \mu, \quad y = \mu^2, \quad z = 0, \quad w = 1, \\ c_2: \quad x &= 0, \quad y = \lambda^2, \quad z = \lambda, \quad w = 1 - \kappa\lambda^2. \end{aligned}$$

The equations of a generator are

$$y - \mu x = \lambda z, \quad \lambda x - \lambda \mu w = (\kappa \lambda^2 - 1) \mu z.$$

The general residual nodal curve is a sextic having four double points at the points of intersection with the two conics. $2c_2^2 + c_{6,2}^2$ (5). The plane $x = 0$ of c_2 cuts c_1 in $\mu = 0$ and in $\mu = \infty$. If λ_1, λ_2 correspond to $\mu = \infty$, and $\lambda_1, \lambda_2, \infty$ are collinear, then $1 + \kappa \lambda_1 \lambda_2 = 0$.

If the (2, 2) correspondence be

$$a\lambda^2\mu^2 + b\lambda^2\mu + c\lambda\mu^2 + d\lambda^2 + e\lambda\mu + f\mu^2 = 0,$$

then $a\lambda^2 + c\lambda + f = 0$ has λ_1, λ_2 for roots. Hence, if $f\kappa + a = 0$, S_6 has a double generator $x = 0, fw - cz = 0$. The residual curve is c_5^2 with two double points on c_1 and four simple points on c_2 . $2c^2 + c_{5,2}^2 + g^2$ (6). Similarly, the plane $z = 0$ cuts c_2 in $\lambda = 0$ and $\lambda = \infty$. When $\lambda = \infty$, the two equations of a generator become identical, but, by eliminating z , we have

$$\mu(y - \mu x)(\kappa \lambda^2 - 1) = \lambda^2(x - \mu w).$$

Hence $\mu_1\mu_2\kappa = 1$. Since μ_1, μ_2 are roots of the equation $a\mu^2 + b\mu + d = 0$, it follows that $f + d = 0$. In this case the surface has a second double generator $z = 0, ay + dw + bx = 0$. The residual is a c_4^2 of the second kind. $2c_2^2 + c_4^2 + 2g^2$ (7). By writing $-ay - bx + cz - dw \equiv u$, the equation of the scroll may be written

$$\begin{vmatrix} az^2 & z(ay - dz) & d(wy - x^2 + z^2) & -dz^2y \\ 0 & 0 & y(u + ay) & dwy - z(cy + ex) \\ u & cy + ex & -dy & 0 \\ 0 & u & -(cy + ex) & dyz \end{vmatrix} = 0.$$

The residual curve in the plane $u - mz = 0$ is a trinodal quartic. By properly choosing m , values of a, b, c, d, e can be found for which this quartic is a double conic. The residual nodal curve is then another c_2 in the plane. $x + m'(dw + cz) = 0$. $4c_2^2 + 2g^2$ (8). This type is self-dual. The new conics have one point in common with each other and establish a similar correspondence to that between λ and μ .

§2.— S_6 Having Two Double Cubics.

4. Consider the two cubics

$$\begin{aligned} x &= 1 - \lambda^2, & y &= \lambda(1 - \lambda^2), & z &= 0, & w &= 1, \\ x &= 1 - \mu^2, & y &= 0, & z &= \mu(1 - \mu^2), & w &= 1, \end{aligned}$$

in which $(a + b)\lambda^2\mu^2 - a\lambda^2 - b\mu^2 = 0$.

The point $(1, 0, 0, 1)$ or $(0, 0)$ is a double element, while $(0, 0, 0, 1)$ counts for four simple self-corresponding elements, $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. The residual curve is in general a quartic $2c_{3,2}^2 + c_4^2$ (9), but may consist of two conics by imposing the condition that the line joining λ to μ shall meet a conic. $2c_{3,2}^2 + 2c_2^2$ (10). Exactly the same forms may exist for two twisted cubics which intersect in five points. One point should be taken for a double point and each of the others for simple self-corresponding points (11), (12).

§3.—*Various Forms.*

$$\begin{aligned} 5. \text{ When the } c_3 \quad x &= 1 - \lambda^2, & y &= \lambda(1 - \lambda^2), & z &= 0, & w &= 1, \\ \text{and } c_2 \quad x &= \mu, & y &= 0, & z &= \mu^2 - \mu, & w &= 1, \end{aligned}$$

are connected by the relation

$$\lambda^2(a\mu^2 + b\mu + c) + (\mu - 1)[a'\lambda\mu - c(\mu - 1)] = 0,$$

the resulting S_6 has the symbol $c_{5,3}^2 + c_2^2 + c_{3,2}^2$ (13). In the plane of the conic are two generators which intersect in the node of the cubic. If $a' = 0$ these two generators coincide. The symbol now becomes $c_4^2 + c_2^2 + c_{3,2}^2 + g^2$ (14).

6. Given a unicursal c_4 and a twisted c_3 cutting c_4 in one node N and in two simple points P_1, P_2 . Let the points λ of c_3 and μ of c_4 be in $(2, 1)$ correspondence such that P_1, P_2 are simple self-corresponding points and N a 2, 1 point. The result is an S_6 of symbol $c_3^2 + c_7^2$ (15), c_7 has four double points.

The line P_1P_2 may be $\lambda_1\mu_1$ and also $\lambda_2\mu_2$, making the symbol $c_3^2 + c_6^2 + g^2$ (16). Each of the points in which g^2 cuts c_4 is on c_6 .

7. If c_3 be nodal, c_7 will have a fourfold point at N . (17). The double generator may appear as before. (18). c_6 has a triple point and one other double point.

8. Consider the twisted quintic

$$x = \lambda^3, \quad y = \lambda^4, \quad z = \lambda^5, \quad w = 1,$$

having one double point and two cusps at the origin, and three apparent double points. The curve lies on the quadric cone $y^2 = xz$. The equations of a line joining the point λ to the point μ may be written

$$\begin{aligned} z - \lambda y &= \mu(y - \lambda x), \\ x(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3) - y(\lambda^2 + \lambda\mu + \mu^2) &= \lambda^3\mu^3w. \end{aligned}$$

If $\lambda = \alpha\mu$, this line will describe an S_6 having c_5 as a double curve. The residual is another c_5 having a triple point at $(0, 0, 0, 1)$. When $\alpha = 1$, the surface reduces to the developable enveloped by the plane

$$\lambda^5w + 10\lambda^2x - 15\lambda y + 6z = 0.$$

This surface is not mentioned in published lists of sextic developables. $2c_{5,3}$ (19). Similarly, by making the same correspondence between the points of a c_5 with two nodes, which are the double points of the projective relation between λ and μ , another S_6 is defined, having two double quintics, each of which has two double points through which the other curve passes. $2c_{5,2}^2$ (20). A particular case is the known developable having a quintic for cuspidal edge. (Schwarz.)

§4.— S_6 Having a Triple Cubic.

9. Given the twisted cubic $x = \lambda, y = \lambda^2, z = \lambda^3, w = 1$. The equations of a line g connecting the point λ to the point μ are

$$x(\lambda + \mu) - \lambda\mu w = y, \quad y(\lambda + \mu) - \lambda\mu x = z,$$

from which

$$\lambda + \mu = \frac{xy - zw}{x^2 - yw}, \quad \lambda\mu = \frac{y^2 - zx}{x^2 - yw}.$$

If λ, μ be pairs of points in a quadratic involution, then g will generate a hyperboloid containing c_3 as a simple curve. Every generator of the same system as g will cut c_3 in two points which belong to the involution. Two generators will touch c_3 ; they correspond to the double points of the involution.

If λ, μ form a [2] involution, g will describe an S_4 having c_3 as double curve. If the [2] involution be cubic, S_4 has a simple rectilinear directrix. Planes through c_1 cut c_3 in a triad defined by the cubic involution. The double points of the involution are the pinch points of the surface. If the relation between λ and μ be $\lambda = \mu$, the quartic developable results.

An S_6 is defined when λ, μ define a [3] involution, on which c_3 is a triple curve. c_3^3 (21). If a double point exist on the cubic curve defined by $f_3(\lambda + \mu, \lambda\mu) = 0$, S_6 has a double generator and becomes unicursal. $c_3^3 + g^3$ (22). If the [3] involution be quartic, the plane containing two generators will contain a third. The points of c_3 are now arranged in quadruples belonging to the involution. A plane quartic curve cannot lie on the surface. A double generator may exist as before. When the involution is of the form

$$\lambda^2\mu^2 = \kappa(\lambda + \mu)^3$$

the generator becomes cuspidal. An S_6 having three families of plane quartics is also defined by any (1, 2) relation between λ and μ . It always has a double generator.

10. In the same way, $S_{2(n-1)}$ can be generated by means of an involution [n] between λ and μ , having c_3 for an $(n-1)$ -fold curve. The genus is $\frac{(n-2)(n-3)}{2}$, and may be made less by the introduction of double points. Apart from double generators, c_3 is in every case the complete nodal curve. When the [n] involution is an $(n+1)$ -ic, a plane containing two generators will also contain a third; the envelope of these planes is a developable of order $\frac{n(n-1)}{2}$. Scrolls can be generated by similar involutions on every unicursal curve.*

* E. Weyr, "Ueber Flächen sechsten Grades mit einer dreifachen cubischen Curve," Wiener Berichte, Vol. 85 (1882), has treated the case of a quartic involution synthetically. The condition for a cubic involution is given in Salmon's *Geometry*, 4th edition, p. 518. The S_6 defined by the quartic involution is self-dual, but not those defined by the general [3] involution.

§5.— S_6 Having a Double Quartic.

11. Various forms of the surface have already been found having a nodal quartic curve; the systematic discussion of all such surfaces will now be given. Let a rational c_4 , depending on a parameter λ , be given. If λ, μ two points on the curve be connected by a $(1, 1)$ correspondence, the lines joining λ to μ will define a rational S_6 , having c_4 for a double curve. The curve will be double, for if $\lambda(T)\mu$, then at any point K there is a point L such that $L(T)K$ and a point M such that $K(T)M$. To find the order of the surface, two cases will be considered, according as c_4 has or has not a node.

Let P be a node on c_4 . There are two values of the parameter at this point, and to each correspond two generators, hence through P pass in general four generators. Project all the points of c_4 from P . The joining line will form a quadric cone K_2 . The lines joining corresponding points will project into planes passing through P . Now cut K_2 by any plane π . The section of K_2 on π will be a conic, and the generators will project into lines connecting corresponding points of two projective ranges on c_2 . These lines envelope another conic, hence all the generators of our scroll touch a quadric cone. The order of a scroll is determined by the number of generators which cut an arbitrary line. A line l through P will cut four generators at P . Two lines tangent to c'_2 in π pass through the trace of l in π , hence l cuts six generators. Since the S_6 is such that every generator touches a quadric cone, it must be of the $(2, 4)$ type. The point P is a sixfold point on the nodal curve, hence the residual is a c_6 having a fourfold point at P , and four simple points on c_4 . $c_{4,2}^2 + c_{6,4}^2$ (23). c_6 is unicursal. If $\lambda = \mu$, the surface becomes the developable on $c_{4,2}^2$.

12. If c_4 is of the second kind, project it from any point P upon it. The result is a K_3 . Through P pass two generators to the scroll. A plane section will cut K_3 in a nodal cubic and the lines which are the projections of the generators will connect corresponding points. Let $z = 0$ be the plane of c_3 ; its equations are

$$x = \lambda^2 (\lambda - 1), \quad y = \lambda (\lambda - 1)^2, \quad w = 1.$$

If $\lambda = \frac{\alpha\mu + \beta}{\gamma\mu + \delta}$ be substituted in the equation of a line joining λ to μ , the result is a quartic in μ . Let x_1, y_1, w_1 be the point in which l pierces $z = 0$. There are four lines through this point which belong to the projection of the system.

Through no point of c_4 can pass more than three generators. At such a point, two of the generators belong to that point, while the third is a trisecant having the other two points for corresponding points. There are four such points on c_4 , hence it follows that the residual nodal curve is a (possibly reducible) c_6 having four double points on c_4 . It is unicursal. The four trisecants of c_4 which are generators to S_6 lie on the S_2 on which c_4 lies. S_6 and H_2 intersect in c_4 and these four generators. $c_4^2 + c_{6,2}^2$ (24). When in particular $\lambda = \mu$, the surface reduces to the developable on the rational quartic and the double points on the nodal sextic become cusps. The reciprocal is the sextic developable on a c_6 having four cusps and six apparent double points.

13. The scrolls having $c_{4,2}^2$ and a composite nodal sextic will first be considered. That in which $c_{6,4}^2$ becomes $2c_{3,2}^2$ was given as (9). Suppose the parameter has the values 0 and 1 at the node. If $\lambda = \kappa$, $\mu = 0$, $\mu = \kappa$, $\lambda = 1$ be two pairs of corresponding elements, the line joining the node to the point κ is a double generator. The symbol now becomes $c_{4,2}^2 + c_{6,3}^2 + g^2$ (25). In particular, the surface may become the developable on $c_{5,3}$ and having one double generator. The $c_{5,3}$ may break up into a c_2 and $c_{3,2}$, the node of $c_{3,2}$ lying on c_2 . Establish a (2, 2) correspondence between c_2 , $c_{3,2}$ such that λ (of c_2) = 0, $\mu = 1, 0$ at the node, and intersecting each other again at $\lambda = 1$, $\mu = \kappa$. Let (0, 0) be a double self-corresponding point and (1, 1), (1, κ) be simple points. The double generator is in the plane of the conic. The symbol is $c_{4,2}^2 + c_{3,2}^3 + c_2^2 + g^2$ (26). If $(\kappa, 0)$, (1, κ); (l , 1), (0, l) be two pairs of corresponding elements and $2\kappa - l - \kappa l = 0$, two double generators pass through the node and the residual is another $c_{4,2}^2$, having its node at the same point. $2c_{4,2}^2 + 2g^2$ (27). This surface is self-dual. If, in particular, $l = \kappa = -1$, the two double generators become tacnodal and the second $c_{4,2}$ breaks up into $2c_2$ which touch. This surface, which is also self-dual, has already been given (4).

14. If the c_4 has a cusp, the resulting S_6 must have a double generator passing through it. The general cuspidal quartic may be written $x = \lambda^2$, $y = \lambda^3$, $z = \lambda^4$, $w = 1$. A line joining λ to $\frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}$ has the equations

$$\begin{aligned} [\lambda^2(\gamma\lambda + \delta)^2 + (\alpha\lambda + \beta)^2]x - \lambda^2(\alpha\lambda + \beta)^2w &= z(\gamma\lambda + \delta)^2, \\ [\lambda(\gamma\lambda + \delta) + (\alpha\lambda + \beta)]y - \lambda(\alpha\lambda + \beta)x &= z(\gamma\lambda + \delta). \end{aligned}$$

Two forms may exist; the general case in which the residual is $c_{5,3}^2$ (28) and that in which $c_{5,3}^2$ breaks up into $c_{3,2}^2 + c_2^2$ (29). If $\beta = 0$, the scroll becomes a quintic. If $\alpha = \delta = 0$ and $\beta = \gamma$, c_4 is simple. The scroll has $x = 0$, $z + w = 0$ for double directrix, $x = 0$, $y = 0$ for triple generator, and $y = 0$, $x + z = 0$ for fourfold directrix. If $\lambda = \mu$, the ordinary quintic developable results.

Any [2] involution will also describe an S_6 of form (28). The involution may be cubical. If $\lambda = 0$, $\mu = 0$ is a pair of corresponding values, the surface reduces to a quintic.

Let $\lambda + \mu = \alpha$, $\lambda\mu = \beta$.

The equations of a generator are

$$(\alpha^2 - \beta)x = \beta^2w + \alpha y, \quad z + \beta x = \alpha y.$$

A [2] involution is of the form

$$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0,$$

hence

$$\alpha = \frac{\Phi_2(t)}{\Psi_2(t)}, \quad \beta = \frac{\chi_2(t)}{\Psi_2(t)},$$

from which a generator has a [2, 4] type. The involution will be cubical if

$$a(a + 2f) - 4hg + bc = 0.$$

Any plane containing two generators will, in this case, contain a third one, hence an infinite number of nodal cubics lie on the surface.

15. When c_4^2 has three apparent double points, $c_{6,2}^2$ may break up into two twisted cubics; this was (11). If a c_4 be chosen such that two trisecants cut the curve in a point having a singular osculating plane, the c_6 reduces to a c_2^3 (30). If $\lambda = \mu$, the trisecants become (ordinary) tangents and the surface is developable. (Rohn's, No. 6.)* In the general case, a trisecant may become a double generator in the same manner as for the nodal quartic. The residual is a c_2^2 , having two double points (31). The limiting case $\lambda = \mu$ is possible when c_4 has

* K. Rohn, "Die rationale Raumcurve 4. Ordnung, zweiter Species." Explanatory memoir to the models of series XXI, Schilling's catalogue of mathematical models (1892).

an inflexional tangent. The scroll becomes the developable with one double generator. The residual c_5 may break up. Given a c_3 and a c_2 having three points in common. Establish a (2, 2) correspondence between them, having two simple self-corresponding points and one double element. The residual is a c_5 having two double points (32). Finally, if A on c_3 corresponds to B on c_2 and A on c_2 to B on c_3 , the line AB is a double generator. The residual is c_4^2 , through AB . $g^2 + c_4^2 + c_3^2 + c_2^2$ (33).

Two trisecants may be double generators. The residual is another c_4^2 of the same form as the first. This surface is self-dual, $2c_4^2 + 2g^2$ (34). If the trisecants, which count as double generators are inflexional tangents and $\lambda = \mu$, the developable with two stationary generators, is the result. This surface is still self-dual. The two forms in which c_4 breaks up into two c_2 have already been given. (2) and (7).

16. The forms of S_6 exist, having a c_4 of the first kind for nodal curve. Let S_2 be a hyperboloid containing c_4 . Every generator g of S_2 cuts c_4 in two points P_1, P_2 . Express a quadratic involution between g, g' of S_2 , the latter cutting c_4 in P'_1, P'_2 . Connect each of the points P_i with each of the points P'_k . The scroll thus described will be order 8 and genus 2, but the order may be reduced to 6 by taking two tangents to c_4 for the double elements of the involution. The (2, 2) involution is elliptic and c_4 is elliptic, hence the residual nodal curve is another c_4 of the same form as the given one. Neither c_4 can break up. $2c_4^2$ (35).

§6.— S_6 Having a Tacnodal Curve.

17. Analogous to the surfaces having a tacnodal straight line, various forms of sextic scrolls exist having one or more tacnodal curves. The former were characterized by the fact that the plane containing two generators through a point on the double line will also contain the line itself. In the latter case the plane of two generators which pass through a point on a nodal curve must contain the tangent to the curve at that point.

Given a conic c_2 in the plane π , and having the parameter λ ; also given a point P not in π . Construct the quadric cone K_2 from P to c_2 . Pass a plane p through P , tangent to K_2 and in it take a conic c'_2 not passing through P . Each tangent plane of K_2 will cut c'_2 in two points P_1, P_2 . Connect the point λ with P_1 and with P_2 . The conic c'_2 will cut π in two points, through each of which

passes one generator lying in π . The plane p cuts c_2 in two consecutive points Q_1, Q_2 . The tangent line at Q_1 and the point P determine a plane which cuts c'_2 in two points collinear with Q_1P . Hence Q_1P is a double generator and Q_2P is a double generator. c'_2 is simple because apart from p only one tangent plane can be drawn to K_2 through the line connecting any point of c'_2 with P . The surface is a rational S_6 . The residual nodal curve is a nodal c_4 . $2\overline{c_2^2} + c_{4,2}^2 + \overline{2g^2}$ (36). If a plane section through the double generator reduces to a perfect square, the $c_{4,2}$ breaks up into $2c_2^2$. $\overline{2c_2^2} + 2c_2^2 + \overline{2g^2}$ (37).

Analytically, the following procedure may be adopted: Let c_2 be defined by $x = \lambda, y = \lambda^2, z = 0, w = 1$, and P be $(0, 0, 1, 0)$. The tangent to c_2 at λ and, passing through P is $\lambda^2w - 2\lambda x + y = 0$. Let $y = 0$ be the equation of p , and let c'_2 be defined by $\phi_2(x, 0, z, w) = 0$. If the tangent plane cuts c'_2 in $x_1, 0, z_1, w_1$, then $y_1 = 0, \phi_2(x_1, 0, z_1, w_1) = 0, \lambda^2w_1 - 2\lambda x_1 = 0$. These equations and those of a line joining λ to $(x_1, 0, z_1, w_1)$ are sufficient to eliminate λ, x_1, z_1, w_1 . The result is the desired S_6 . If c_2, c'_2 have a point in common the surface will be a quintic, on which c'_2 is a simple conic and c_2 a tacnodal conic. If c_2, c'_2 touch each other, the surface becomes a quadric cone counted twice.

18. Now suppose p is not tangent to K_2 . If c_2, c'_2 intersect in two points, the same construction as before will generate a sextic, but c'_2 will be double since two tangent planes to K_2 can be drawn through any line in its plane. As a $(2, 2)$ correspondence between c_2, c'_2 is defined, without any double element, the scroll will be elliptic. The residual nodal curve is a non-singular cubic. $[c_3^2] + c_2^2 + \overline{2c_2^2}$ (38). In the plane of the tacnodal conic the simple generators are tangents at the intersections of c_2 and c'_2 .

19. Given a non-singular $[c_3]$ lying in $w = 0$ and cutting c_2 in one point. Lines joining corresponding points as defined in 17 will generate an $S_6, p = 1$ of symbol $c_6^2 + \overline{2c_2^2}$ (39). Let $[c_3]$ be defined by $z^2y = x(ax^2 + bxy + cy^2)$. Since $2x_1\lambda = y_1$, from the equations of a generator we easily obtain

$$\begin{vmatrix} 2bw & f & g & ax^2 & 0 \\ 4cw & w(2bw - 8cx) & f + 4cyw & g & ax^2 \\ 1 & -2x & y & 0 & 0 \\ 0 & w & -2x & y & 0 \\ 0 & 0 & w & -2x & y \end{vmatrix} = 0$$

as the equation of the scroll; $f = 4cx^2 - 4bxw + aw^2 - 4cyw$, $g = 2bx^2 - 2axw - 2z^2$. If $[c_3]$ be not in a tangent plane to K_2 , but cut c_2 in two points, the tangent at each passing through P , (38) would result. The three generators in $w = 0$ coincide with $x = 0$, $w = 0$.

20. Let c_3 be a nodal cubic in $w = 0$, so that $b^2 - 4ac = 0$. The line joining the node to the corresponding λ is a double generator. $c_{5,3}^2 + 2c_2^2 + g^2$ (40). When g^2 does not lie in the plane of c_2 , c_5 must have a threefold point at its intersection with c_2 . When g^2 lies in the plane of c_2 , it is a tangent to the latter.

21. Suppose P does not lie in the plane of c'_2 , but that the latter has one point in common with c_2 . In order for the surface to be a sextic, P must lie in the common tangent plane at the point of intersection of c_2 , c'_2 . Both conics are double and the surface is rational. The point P is so chosen that from it c_2 , c'_2 project into cones having double contact. In the second common tangent plane is a g^2 , and the residual nodal curve is a c_3^2 . $c_2^2 + 2c_2'^2 + c_3^2 + g^2$ (41). If c'_2 touch c_2 , but not lie in a plane through P , the common tangent is a g^2 . The residual curve is a plane nodal cubic having its node at the point of contact of c_2 , c'_2 . $c_2^2 + 2c_2'^2 + c_{3,2}^2 + g_2$ (42). This form can also be generated by establishing a similar correspondence between c_2 and $c_{3,2}$.

22. Given a c_3 , $x = \lambda$, $y = \lambda^2$, $z = \lambda^3$, $w = 1$. The equation of a plane t containing the tangent to c_3 at λ is

$$\lambda^2 w - 2\lambda x + x(\lambda^2 x - 2\lambda y + z) = 0.$$

Given a conic c_2 , defined by $ax + by + cz + dw = 0$, $\phi_2(x, y, z, w) = 0$, such that c_2 has three points on c_3 . The plane t will cut c_2 in two points x_1, y_1, z_1, w_1 . Join each such point with λ . By substituting the coordinates of any point of c_2 in t , it becomes a quadratic in λ , hence c_2 is a double conic. Let A, B, C be the points common to c_2, c_3 , let the tangent plane t contain the tangent to c_2 at A , and let $t(A)$ be a perfect square in λ . A is then a double point in the (2, 2) correspondence between c_2 and c_3 . B and C are simple self-corresponding points. The residual nodal curve is another conic having one point of intersection with c_2 . $2\overline{c_3^2} + 2c_2^2$ (43).

If c_2 touches c_3 and has one other point on it, an S_6 , having c_2 for a tacnodal curve, may be described as follows: Let P be a fixed point on c_3 . Draw the tangent planes from P to c_2 . Connect the point of contact with each of the remaining points of intersection with c_3 . As before, c_3 will be double. $\overline{2c_2^2} + 2c_3^2$ (44). The common tangent is a torsal generator.

23. Given

$$\begin{aligned} c_1: \quad x &= \lambda, \quad y = \lambda^2, \quad z = 0, \quad w = 1, \\ c_2: \quad x &= 0, \quad y = \mu^2, \quad z = \mu, \quad w = 1. \end{aligned}$$

The equation of a plane containing a generator is of the form

$$\mu x + \lambda z - \lambda \mu w + m(\lambda x - y + \mu z) = 0.$$

The equation of a plane containing the tangent to c_1 at λ is

$$2\lambda x - \lambda^2 w - y + \lambda z = 0.$$

When these two planes coincide, the two generators through each point λ of c_1 lie in a plane containing the tangent. The resulting equation is

$$2\lambda \mu x + (\lambda^2 + \mu^2) z - \lambda^2 \mu w - \mu y = 0.$$

If a (3, 2) correspondence between λ, μ , having (∞, ∞) and $(0, 0)$ as double elements be given, the line will generate an S_6 of symbol $\overline{2c_2^2} + c_3^2$ (45). If $\lambda^3 - \mu^2 - \lambda^2 = 0$, the equation becomes

$$\begin{vmatrix} wy - z^2 & xy + yw - x^2 - z^2 & 0 \\ wx & x^2 - z^2 + yw & xy \\ wx & x^2 & x^2 + z^2 - yw \end{vmatrix} = 0.$$

The generators in the plane $z = 0$ are $w = 0$ and $x = y$.

If between λ, μ a general (2, 1) relation exists, the result is an S_6 of symbol $c_6^2 + \overline{2c_2^2}$ (46). Similarly, if between a c_3 and a c_2 , with two points on c_3 , be put in (2, 1) correspondence such that each point of intersection is a simple self-corresponding point, the result has the symbol $c_4^2 + \overline{2c_3^2}$ (47).

24. Given c_3 , $x = \lambda$, $y = \lambda^2$, $z = \lambda^3$, $w = 1$ and c_2 , $x = y = \mu$, $z = \mu^2$, $w = 1$, cutting c_3 in three points $\lambda = \mu = 0, 1, \infty$. Write the condition that the plane containing a generator from λ will also contain the tangent to c_3 at λ . Impose the condition that the same plane contains the tangent to c_2 at μ . Finally, make $(0, 0)$ a double element and $(1, 1)$, (∞, ∞) each simple self-corresponding elements in a $(2, 2)$ correspondence between λ, μ . The resulting S_6 has the symbol $\overline{2c_2^3} + \overline{2c_3^2}$ (48).

25. Given a $c_{5,3}$. Establish a projective correspondence between two points on the curve in such a way that the two generators passing through any point of the curve lie in a plane containing the tangent to the curve at that point. Two of the values of the parameter at the triple point should be corresponding points of the projective correspondence. $\overline{2c_{5,3}^2}$ (49). If instead of a $c_{5,3}$ a rational c_5 with two double points be given, and upon it a $(1, 1)$ correspondence with one value of the parameter at each node as self-corresponding element, the resulting scroll will have this $\overline{2c_{5,2}^2}$ as tacnodal curve (50).

26. Given c_2 ; $x = \lambda^2$, $y = \lambda$, $z = 0$, $w = 1$ and $[c_3]$ defined by

$$y = 0, xz(a'z + a''w) + zw(bx + b'z) + wx(cx + c'z + c''w) = 0.$$

Describe a K_2 from $(0, 0, 1, 0)$ to c_2 . The tangent plane $\lambda^2 w + x - 2\lambda y = 0$ will cut $[c_3]$ in $(x_1, 0, z_1, w_1)$, $[c_3]$ will be a double curve. By letting $d = a'' + b + c'$, and writing $a'z^2 - dzw + cwx - cy^2 + c''w^2 \equiv \phi$, the equation becomes

$$\begin{vmatrix} \phi & y(dz - 2c''w) & c''y^2 - b'z^2 & 0 \\ 0 & \phi & y(dz - 2c''w) & c''y^2 - b'z^2 \\ w & -2y & x & 0 \\ 0 & w & -2y & x \end{vmatrix} = 0.$$

This is the equation of (38). If $c = 0$, c_3 has a node at $(1, 0, 0, 0)$ and $w = 0$, $z = 0$ is a g^2 . The form is now (42), and its reciprocal is (37). A plane $x = \alpha w$ will have in each case a singularity of the same form at $(\alpha, -\alpha, 0, 1)$ as at $(\alpha, \alpha, 0, 1)$. If the condition be imposed that the intersection of this plane be an oscnode, the other c_2^2 of (38) will be consecutive to the given one. The tacnodal tangent of any plane section is the tangent to K_2 in that plane. The conic

will be oscnodal in each of the three forms mentioned. Their symbols become:

$$\overline{3c_2^2} + [c_3^2] \quad (51), \quad \overline{3c_2^2} + c_{3,2}^2 + g^2 \quad (52), \quad \overline{3c_2^2} + c_2^2 + 2\overline{g^2} \quad (53).$$

The last form can also be derived as a limiting case of (3).

§7.—*Corrections to Previous List.*

27. The surface of genus 3 cannot have a nodal curve of order 7, so that type I, $p = 3$, c_7^2 is impossible. (Journal, Vol. XXV, p. 267). Wiman* proved that for every scroll not contained in a linear congruence $6p \leq (n-2)(n-3)$. Hence, when $n = 6$, $p \succ 2$.

I assumed that two plane non-singular quartics could be found which are in (1, 1) point correspondence without being projective. Given two non-singular plane quartics c in plane π and c' in π' , between which exists a (1, 1) point correspondence. Superpose the curves in such a way that the point A is self-corresponding, and that B, B' , two other corresponding points, are collinear with A . This is no restriction on either curve. Now construct another curve κ projective with c' , such that $A = A, B$ goes into B' . Between c, κ exists a (1, 1) correspondence with two self-corresponding points of intersection, hence lines joining corresponding points will generate a sextic scroll of genus 3. From the preceding theorem a sextic scroll of genus 3 must belong to a linear congruence. A quartic curve can lie on such a scroll only when the residual section is a double generator. Hence the line AB is a double generator. Generators were defined as lines joining corresponding points. If AB cuts c in C, D , and κ in C', D' , it follows that c corresponds to c' (or D'), and that D corresponds to D' (or C'). When a (1, 1) correspondence between the points of a line exists, the correspondence must be projective. By properly choosing A, B any line in the plane can be taken for AB , hence c, κ are projective. Since c', κ are projective, c, c' are projective. The same reasoning can be applied to non-singular plane curves of any order, hence: *when a (1, 1) correspondence can be established between the points of two non-singular plane algebraic curves, the curves are projectively equivalent.*†

* Klassifikation af regelytorna af sjette graden. Lund (Dissertation) 1892.

† This theorem was proved for cubic curves by Schwarz, "Ueber die geradlinigen Flächen fünften Grades," *Crelle*, Vol. 64, p. 27. For curves of order > 4 probably (1, 1) correspondence other than projectivity can exist only for much lower genus than the maximum.

§—8. *Table of New Results.*

1	$2c_2^2 + c_4^2 + g^2$	$p = 1$	28	$c_{4,2}^2 + c_{5,3}^2 + g^2$	2
2	$2c_2^2 + c_4^2 + 2g^2$	3	29	$c_{4,2}^2 + c_{3,2}^2 + c_2^2$	2
3	$4c_2^2 + \overline{2g^2}$	2	30	$c_4^2 + c_2^2$	3
4	$2c_2^2 + c_{4,2}^2 + \overline{2g^2}$	2	31	$c_4^2 + c_{5,2}^2 + g^2$	3
5	$2c_2^2 + c_{6,2}^2$	3	32	$c_3^2 + c_2^2 + c_{5,2}^2$	3
6	$2c_2^2 + c_{5,2}^2 + g^2$	3	33	$c_4^2 + c_3^2 + c_2^2 + g^2$	3
7	$2c_2^2 + c_4^2 + 2g^2$	3	34	$2c_4^2 + 2g^2$	3
8	$4c_2^2 + 2g^2$	3	35	$2c_4^2$ (first kind)	$p = 2$
9	$2c_{3,2}^2 + c_{4,2}^2$	2	36	$\overline{2c_2^2} + c_{4,2}^2 + \overline{2g^2}$	2
10	$2c_{3,2}^2 + 2c_2^2$	2	37	$\overline{2c_2^2} + 2c_2^2 + \overline{2g^2}$	2
11	$2c_3^2 + c_4^2$	3	38	$[c_3^2] + c_2^2 + \overline{2c_2^2}$	$p = 1$
12	$2c_3^2 + 2c_2^2$	3	39	$\overline{2c_2^2} + c_{5,2}^2$	$p = 1$
13	$c_{5,3}^2 + c_2^2 + c_{3,2}^2$	2	40	$\overline{2c_2^2} + c_{5,3}^2 + g^2$	2
14	$c_{4,2}^2 + c_2^2 + c_{3,2}^2 + g^2$	2	41	$\overline{2c_2^2} + c_3^2 + c_2^2 + g^2$	3
15	$c_3^2 + c_{7,2}^2$	3	42	$\overline{2c_2^2} + c_{3,2}^2 + c_2^2 + g^2$	2
16	$c_3^2 + c_{6,2}^2 + g^2$	3	43	$\overline{2c_3^2} + 2c_2^2$	3
17	$c_{3,2}^2 + c_{7,4}^2$	2	44	$\overline{2c_2^2} + 2c_3^2$	3
18	$c_{3,2}^2 + c_{6,3}^2 + g^2$	2	45	$\overline{2c_2^2} + c_2^2$	3
19	$2c_{5,3}^2$	2	46	$\overline{2c_2^2} + c_{6,4}^2$	2
20	$2c_{5,2}^2$	3	47	$\overline{2c_3^2} + c_4^2$	3
21	c_3^2	$p = 1$	48	$\overline{2c_2^2} + \overline{2c_3^2}$	3
22	$c_3^2 + g^2$	3	49	$\overline{2c_{5,3}}$	2
23	$c_{4,2}^2 + c_{6,4}^2$	2	50	$\overline{2c_{5,2}}$	3
24	$c_4^2 + c_{6,2}^2$	2	51	$\overline{3c_2^2} + [c_3^2]$	$p = 1$
25	$c_{4,2}^2 + c_{5,3}^2 + g^2$	2	52	$\overline{3c_2^2} + c_{3,2}^2 + g^2$	2
26	$c_{4,2}^2 + c_{3,2}^2 + c_2^2 + g^2$	2	53	$\overline{3c_2^2} + c_2^2 + \overline{2g^2}$	3
27	$2c_{4,2}^2 + 2g^2$	2			